MATH2050C Selected Solution to Assignment 10

2, 3, 4, 5, 6, 12, 13, 15, 17.

Section 5.3

(1) By Max-min Theorem, f attains its minimum at some $z \in [a, b]$ and $f(z) > 0$ by assumption. It follows that $f(x) \geq f(z) > 0$ for all $x \in [a, b]$.

(3) Define a sequence $\{x_n\}$ be $|f(x_{n+1})| \leq |f(x_n)|/2$ where $x_1 \in [a, b]$ is arbitrary. We have $|f(x_n)| \leq |f(x_1)|/2^{n-1}$, and so $\lim_{n\to\infty} |f(x_n)| \leq \lim_{n\to\infty} |f(x_1)|2^{1-n} = 0$. By Bolzano-Weierstrass, there is a subsequence $\{x_{n_j}\}$ converging to some $z \in [a, b]$. By continuity, $f(z) = \lim_{j\to\infty} f(x_{n_j}) = 0$. (Note that $\{a_n\}$ tends to 0 if and only if $\{|a_n|\}$ tends to 0.)

(4) For a polynomial p of odd degree, $p(x)$ at $\pm\infty$ must be of different sign. Hence, we can find a large $x > 0$ and a large $y < 0$ such $f(x)f(y) < 0$. Applying Root Theorem to p on $[y, x]$ we find some $c \in [y, x]$ such that $p(c) = 0$.

(5) $p(-10) = 2991, p(0) = -9$, and $p(2) = 63$. By the theorem on Existence of Zeros, there is a zero in $(-10, 0)$ and another in $(0, 2)$.

(6) The function g satisfies $g(0) = f(0) - f(1/2)$ and $g(1/2) = f(1/2) - f(1) = f(1/2) - f(0) = -g(0)$. It is also continuous on $[0, 1/2]$. If $g(0) = 0$, we are done. If $g(0) \neq 0, g(0)g(1/2) = -g(0)^2 < 0$, so the desired conclusion comes from the theorem on Existence of Zeros.

Note. An advanced result called the Borsuk-Ulam Theorem asserts that any continuous mapping F from the unit sphere

$$S = \left\{x \in \mathbb{R}^n : \ x_1^2 + x_2^2 + \cdots + x_n^2 = 1\right\},$$

to $\mathbb{R}^n$ satisfies the following property: There exists a point $p \in S$ so that $F(p) = F(-p)$. This exercise is essentially the case $n = 1$.

(12) The function $g(x) = \cos x - x^2$ satisfies $g(0) = 1 > 0$ and $g(\pi/2) < 0$, so there is some $x_0 \in (0, \pi/2)$ such that $g(x_0) = 0$. Since $\cos x$ is strictly decreasing and x^2 is strictly increasing on $[0, \pi/2]$, g is strictly decreasing and x_0 is the unique zero for g. It means $g(x) > 0$, that is, $\cos x > x^2$ on $[0, x_0)$ and $g(x) < 0$, that is, $\cos x < x^2$ on $(x_0, \pi/2]$. It implies $f(x) = \cos x$ on $[0, x_0)$ and $f(x) = x^2$ on $(x_0, \pi/2]$. The conclusion comes from the fact that $\cos x > \cos x_0$ on $[0, x_0)$ and $x^2 > x_0^2$ on $(x_0, \pi/2]$.

(13) As $f \to 0$ as $x \to \infty$, for $\varepsilon = 1$, there is some M such that $|f(x) - 0| < 1$ for all $x, x \geq M$. Similarly, there is some N such that $|f(x) - 0| < 1$ for all $x, x < -N$. On the other hand, by the Boundedness Theorem, there is some L such that $|f(x)| \leq L$ for $x \in [N, M]$. We conclude that $|f(x)| \leq \max\{1, L\}$.

In case $f > 0$ somewhere, say, $f(z) > 0$ for some z. Let $\varepsilon = f(z)/2 > 0$, we find K such that $|f(x) - 0| < f(z)/2$ for all $x \in (-\infty, -K) \cup (K, \infty)$. So the supremum of f over $\mathbb{R}$ is equal to the supremum of f over $[-K, K]$. Now by the Max-min Theorem, we conclude the minimum is attained on $[-K, K]$. When $f < 0$ everywhere, consider $-f$.

The function $f(x) = e^{-x^2}$ attains its maximum at $x = 0$ but its infimum, 0, is never attained.

(15) $f(x) = x^2$ is increasing on $[0, \infty)$ hence any open (resp. closed) subinterval in $[0, \infty)$ is mapped onto an open (resp. closed) interval. Similarly, the function is decreasing on $(-\infty, 0]$, hence any open (resp. closed) subinterval in $(-\infty, 0]$ is mapped onto an open (resp. closed) interval. On the other hand, whenever (a, b) contains the origin, since $f(0) = 0$ is the minimum, the image of $f((a, b))$ is of the form $[0, c)$ for some positive c.

(17) Yes, f must be a constant function. Suppose not, there are rational numbers $r_1, r_2, r_1 < r_2$, such that $f(x) = r_1, f(y) = r_2$. Pick an irrational number h between r_1 and r_2. Bolzano's Theorem asserts that $f(z) = h$ for some z, contradicting the assumption on f.

Supplementary Problems

1. Let $f \in C[a, b]$. Suppose that $f(x) > 0$ for all $x \in [a, b]$. Show that there is some $\rho > 0$ such that $f(x) \geq \rho$ for all $x \in [a, b]$. Hint: Use Suppl. Problem no 3 in Ex 9 and apply the Heine-Borel Theorem.

   Solution At each $x \in [a, b], f(x) > 0$. Hence there is some $\delta_x > 0$ such that $|f(y) - f(x)| < f(x)/2$ for $y \in I_x \equiv (x - \delta_x, x + \delta_x)$. It follows that $f(y) > f(x)/2, y \in I_x$. (See Suppl Problem no 3 in Ex 9.) The collection of open intervals $\{I_x\}$ covers $[a, b]$. By Heine-Borel Theorem, there is a finite subcover from $\{I_x\}$. It follows that each $x \in [a, b]$ must belong to some I_{x_i}. Therefore, letting $\rho = \frac{1}{2} \min\{f(x_1), \cdots, f(x_n)\}$, we have $f(x) \geq f(x_i)/2 \geq \rho > 0$.

2. Use the previous problem to deduce the Max-Min Theorem.

   Solution First of all, by Boundedness Theorem f is bounded so that $\alpha \equiv \inf f$ is a finite number. Consider the continuous function $g(x) = f(x) - \alpha$. When the minimum is not attained, $g(x) > 0$ everywhere. By the previous problem, there is some $\rho > 0$ such that $g(x) \geq \rho$, that is, $f(x) \geq \inf f + \rho$, contradicting the definition of infimum.

3. Let A be a set in $\mathbb{R}$ which contains at least two points and satisfies: Whenever $x, y \in A, x < y$, implies the interval $[x, y] \subset A$. Prove that A is an interval.

   Solution Let's assume A is bounded so $a = \inf A$ and $b = \sup A$ are finite. (When A is unbounded on one side or both sides, the proof is similar.) We claim $(a, b) \subset A \subset [a, b]$ and this implies that A is an interval (closed, open, half-open). First of all, the second inclusion $A \subset [a, b]$ follows trivially from $a = \inf A \leq x \leq \sup A = b, x \in A$. On the other hand, let $x \in (a, b)$. From the definition of the infimum and supremum, we can find some $z \in A$ so close to a such that $a < z < x$. Similarly, there is some $w \in A$ so close to b such that $x < w < b$. By assumption $[z, w] \subset A$. In particular, $x \in A$. We have shown that $(a, b) \subset A$.